

Analytic Function

① $\nabla \neq \text{diff}$ لو قايمة للتفاضل عند Z_0 .

② Analytic " " " " " و جوارها

Z-plane ^{has} no " " " \Leftarrow entire (3)

* أي دالة تقصوى على \bar{z} أو $|z|$ غير analytic.

* الدالة (analytic) أو حقيقة شردا Cauchy-Riemann

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

$$U_r = \frac{1}{r} V_\theta$$
$$V_r = -\frac{1}{r} U_\theta$$

$$f(z) = u_x + i v_x \quad \hat{f}(z) = e^{-i\theta} (u_r + i v_r)$$

⑤ * Prove that $\frac{d}{dz}(z^2 \bar{z})$ doesn't exist
any where.

Sol

$$f(z) = z^2 \bar{z} = (x+iy)^2 (x-iy)$$

$$= (x^2 + i2xy - y^2) (x-iy)$$

$$= x^3 + i2x^2y - y^2x - ix^2y + 2xy^2 + iy^3$$

$$= \underbrace{(x^3 + y^2x)}_u + i \underbrace{(x^2y + y^3)}_v$$

$$u_x = 3x^2 + y^2$$

$$u_y = 2xy$$

$$v_x = 2xy$$

$$v_y = x^2 + 3y^2$$

$$u_x \neq v_y, \quad u_y \neq -v_x$$

not analytic \Rightarrow not diff. \neq

[2]

6] Are the following function analytic?

$$a) f(z) = \operatorname{Re}(z^2) = \operatorname{Re}((x+iy)^2)$$

$$= \operatorname{Re}(x^2 + i2xy - y^2)$$

$$f(z) = x^2 - y^2$$

$$u = x^2 - y^2$$

$$v = 0$$

$$u_x = 2x$$

$$u_y = -2y$$

$$v_x = 0$$

$$v_y = 0$$

$$\therefore u_x \neq v_y, \quad u_y \neq -v_x$$

→ not analytic

$$b) f(z) = z - \bar{z} = (x+iy) - x+iy \\ = i2y$$

$$u=0$$

$$u_x=0$$

$$u_y=0$$

$$v=2y$$

$$v_x=0$$

$$v_y=2$$

→ not analytic

$$c) f(z) = z^2 = (x+iy)^2$$

$$= x^2 + i2xy - y^2$$

$$u = x^2 - y^2$$

$$v = 2xy$$

$$u_x = 2x$$

$$u_y = -2y$$

$$v_x = 2y$$

$$v_y = 2x$$

$$u_x = v_y, \quad u_y = -v_x \Rightarrow \text{analytic}$$

$$d) f(z) = \frac{x}{x^2+y^2} + i \frac{y}{x^2+y^2}$$

$$\text{Let } u = \frac{x}{x^2+y^2}, \quad v = \frac{y}{x^2+y^2}$$

$$\cancel{u_x = \frac{(x^2+y^2) - x \cdot 2x}{(x^2+y^2)^2}}$$

$$u_x = \frac{-2xy}{(x^2+y^2)^2}$$

$$u_y = \frac{x^2 - y^2}{(x^2+y^2)^2}$$

$$v_x = \frac{-x^2 + y^2}{(x^2+y^2)^2}$$

$$v_y = \frac{-2xy}{(x^2+y^2)^2}$$

$$u_x = v_y$$

$$u_y = -v_x$$

> analytic

* Let $f(z) = u + iv$ analytic fn

$$u = 3x^2y + 2x^2 - y^3 - 2y^2 \text{ find } f'(z)$$

without $f(z)$ and Conjugate harmonic (v)

Sol

$$f'(z) = u_x + i v_x = u_x - i u_y$$

$$= (6xy + 4x) - i(3x^2 - 3y^2 - 4y)$$

$$u_x = v_y = 6xy + 4x$$

$$v_x = -u_y = -3x^2 + 3y^2 + 4y$$

$$v = \int v_y = 3xy^2 + 4xy + f(x)$$

مع هنا قبل v بالنسبة
 x و v_x و v_y بالنسبة

$$v_x = 3y^2 + 4x + f'(x) = -3x^2 + 3y^2 + 4y$$

$$f'(x) = -3x^2 \Rightarrow f(x) = -x^3 + C$$

$$v = 3xy^2 + 4xy + ~~f(x)~~ - x^3 + C$$

Harmonic fn

$$u_{xx} + u_{yy} = 0, \quad v_{xx} + v_{yy} = 0$$

$$r^2 u_{rr} + r u_r + u_{\theta\theta} = 0$$

تقريب (analytic) و
harmonic

* Prove that $u = \ln(x^2 + y^2)$ harmonic.

$$u_x = \frac{2x}{x^2 + y^2} \rightarrow u_{xx} = \frac{2y^2 - 2x^2}{(x^2 + y^2)^2}$$

$$u_y = \frac{2y}{x^2 + y^2} \rightarrow u_{yy} = \frac{-2x^2 + 2y^2}{(x^2 + y^2)^2}$$

$$u_{xx} + u_{yy} = 0$$

→ harmonic

* determine which the following fn are possible to be real part of analytic fn or not.

a) $u = x^2 + y + 3$

$$u_x = 2x$$

$$u_{xx} = 2$$

$$u_y = 1$$

$$u_{yy} = 0$$

$$u_{xx} + u_{yy} \neq 0$$

→ This function is not harmonic \Rightarrow not analytic

b) $u = \cosh x \cosh y$

$$u_x = \sinh x \cosh y$$

$$u_{xx} = \cosh x \cosh y$$

$$u_y = \cosh x \sinh y$$

$$u_{yy} = \cosh x \cosh y$$

$$u_{xx} \neq u_{yy} \neq 0$$

→ not harmonic \Rightarrow not analytic

* Show that if $f(z) = u + iv$ is analytic

$$\nabla^2 |f(z)|^2 = 4 \left| \frac{df}{dz} \right|^2 \quad \nabla \Rightarrow \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$$

\swarrow \searrow
 $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ $u^2 + v^2$

$$\left| \frac{df}{dz} \right|^2 = u_x^2 + v_x^2$$

$$\frac{\partial}{\partial x} (u^2 + v^2) = 2u u_x + 2v v_x$$

$$\frac{\partial^2}{\partial x^2} (u^2 + v^2) = 2u u_{xx} + 2u_x^2 + 2v v_{xx} + 2v_x^2 \rightarrow (1)$$

$$\frac{\partial^2}{\partial y^2} (u^2 + v^2) = 2u u_{yy} + 2u_y^2 + 2v v_{yy} + 2v_y^2 \rightarrow (2)$$

From (1) + (2)

$$(1) + (2)$$

$$2u (u_{xx} + u_{yy})$$

$$+ 2 (u_x^2 + v_x^2)$$

$$+ 2u (v_{xx} + v_{yy}) + 2 (u_y^2 + v_y^2)$$

Q

~~$$u_{xx} + v_{yy}$$~~

$$u_{xx} + u_{yy} = 0$$

$$v_{xx} + v_{yy} = 0$$

$$u_x^2 + v_x^2 = \tilde{f}(z)$$

$$u_y^2 + v_y^2 = \tilde{f}(z)$$

$$\therefore \textcircled{1} + \textcircled{2}$$

$$= 4 \left| \frac{d\tilde{f}}{dz} \right|^2$$

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